The Mathematical Representation of a Light Pulse.

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In recent years, owing to the work of Rayleigh, Schuster, and others, our views as to the nature of white light have undergone a change, and it is now universally accepted that white light consists of irregular pulses which are transformed into trains of sine waves by their passage through a prism. But the method of this transformation is not clearly understood, as the reasoning on the subject is unfortunately somewhat general, the only concrete case well known being the pulse Rayleigh represented by $e^{-c^2t^2}$. Had the nature of a light pulse been better understood, there might possibly never have been any talk of the "Light Quantum" or unit theory of light.

The object of this note is to call attention to a new class of expressions representing the initial form and dispersion of a light pulse. They are both simple and elegant, and one of them gives the energy distribution required by Wien's law for black body radiation. They have been suggested by one of Kelvin's hydrodynamical papers.* They do not depend on the Fourier analysis and this is an advantage, for we never know how much of the latter is subjective.

Consider the equation

$$\frac{\partial^2 X}{\partial x^2} + c^2 \frac{\partial^4 X}{\partial t^4} = 0. \tag{1}$$

If we substitute $X = \sin \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right)$, we find that $v = \pm \tau/2\pi c$, *i.e.* the equation represents the propagation of waves in a medium in which the velocity is directly proportional to the period. This is the law of deep-sea waves; it is obeyed by no medium for light waves although it gives the variation with the period in the right direction. After the necessary corrections have been made the distribution of the energy in the spectrum of an incandescent solid is independent of the prism producing the spectrum. Results obtained therefore for a hypothetical medium obeying the above law can easily be transferred to any other medium.

Equation (1) can be written

$$\frac{\partial X}{\partial (ix)} = \pm c \frac{\partial^2 X}{\partial t^2}.$$

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[*] "Initiation of Deep-Sea Waves, etc.," 'Proc. Roy. Soc. Edin.,' 1906, vol. 26, p. 399.

This is formally the same as the equation for the one-dimensional conduction of heat, namely,

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}.$$

Kelvin's instantaneous-plane-source solution for the latter is

$$\frac{1}{\sqrt{t}} e^{-x^2/4\kappa t}.$$

By analogy, $$\frac{1}{\sqrt{(ix)}} e^{\mp t^2/4c\,(ix)},$$

is a solution of (1). Disregard the plus sign in the index and add a constant h to ix. The expression still satisfies (1) and becomes

$$\frac{1}{\sqrt{(h+ix)}} e^{-t^2/4c\,(h+ix)}. \tag{2}$$

Since (1) is linear in X, we can differentiate (2) n times with regard to ix and the result,

$$\frac{\partial^n}{\partial\,(ix)^n} \frac{1}{\sqrt{(h+ix)}} e^{-t^2/4c\,(h+ix)}, \tag{3}$$

is still a solution.

To find the initial form of this solution put $t=0$ in (3) and then differentiate n times with respect to ix. The result is

$$(-1)^n \tfrac{1}{2}\tfrac{3}{2}\tfrac{5}{2} \dots (n-\tfrac{1}{2}) \frac{1}{(h+ix)^{n+\frac{1}{2}}}.$$

Write $h+ix=\rho e^{i\theta}$, disregard the constant factor, and this becomes

$$\frac{\{\cos{(n+\frac{1}{2})}\,\theta - i\sin{(n+\frac{1}{2})}\,\theta\}}{\rho^{n+\frac{1}{2}}}.$$

We take the first part of this expression as the initial form, *i.e.* we choose the real part of (3). In fig. (1) the initial form of the disturbance is represented on the same scale for $n = 1, 2, 3, 4,$ and 5.

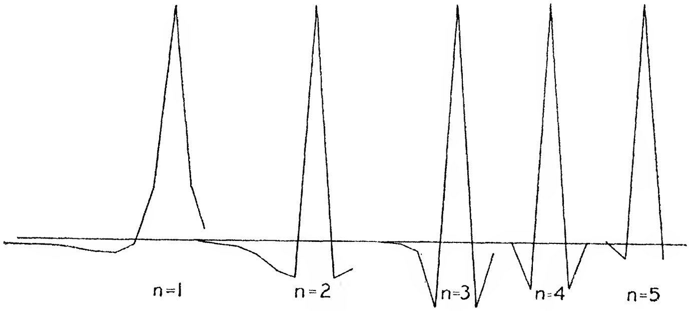

Fig. 1.

Kelvin has already (*loc. cit.*) treated the case of a deep-sea wave having the first of the above initial forms, and has shown how the disturbance spreads within a distance of a few wave-lengths from the origin. But it is here necessary to proceed quite otherwise, since in optics the observer is always at a much greater distance than this from the light source.

The quantity c is a constant for the medium. Let us fix it by supposing that a harmonic wave of wave-length $\lambda = 5 \times 10^{-5}$ cm. travels in the medium with a velocity of 3×10^{10} cm./sec. It has already been shown that the velocity of such a wave is $\tau / 2\pi c$. Hence putting $\tau = 5 \times 10^{-5} / 3 \times 10^{10}$ we obtain

$$3 \times 10^{10} = \frac{5 \times 10^{-5}}{2\pi c\,(3 \times 10^{10})},$$

or
$$c = \frac{5 \times 10^{-5}}{2\pi\,(3 \times 10^{10})^2} = 9 \times 10^{-27} \text{ approximately, in (sec.}^2\text{/cm.).}$$

The initial disturbance is given by

$$\frac{\cos\,(n+\tfrac{1}{2})\,\theta}{\rho^{n+\frac{1}{2}}},$$

hence its value at the origin is $1/h^{n+\frac{1}{2}}$. If a be the abscissa of the point for which it has half its maximum value,

$$\frac{\cos\,(n+\tfrac{1}{2})\,\theta}{(h^2+a^2)^{(2n+1)/4}} = \frac{1}{2h^{n+\frac{1}{2}}},$$

$$(h^2+a^2)^{(2n+1)/4} < 2h^{n+\frac{1}{2}},$$

$$h^2+a^2 < h^2 2^{4/(2n+1)},$$

$$a^2 < h^2\,(2^{4/(2n+1)}-1),$$

a can be taken as a measure of the breadth of the initial disturbance.

We shall examine the disturbance when its maximum has travelled out a distance of 2 cm. from the origin, and shall suppose a small in comparison with 2 cm. Then in general h is small compared with 2 cm. A wave for which $\lambda = 5 \times 10^{-5}$ would take $\frac{2}{3} \times 10^{-10}$ secs. to travel out 2 cm., and an irregular disturbance will take something of the same order. While the disturbance is passing a point 2 cm. out, t will be about $\frac{2}{3} \times 10^{-10}$ secs., and consequently t^2/c approximately

$$\frac{(\frac{2}{3} \times 10^{-10})^2}{9 \times 10^{-27}} = 5 \times 10^5, \text{ in cms.}$$

The fact that t^2/c is so large simplifies things very considerably. In (3) the only term to be retained is that of the highest power in t^2/c, which is easily seen to be

$$\frac{1}{\sqrt{(h+ix)}} \left\{ \frac{t^2}{4c\,(h+ix)^2} \right\}^n e^{-t^2/4c\,(h+ix)} = \frac{t^{2n}}{4^n c^n \rho^{2n+\frac{1}{2}} e^{i\,(2n+\frac{1}{2})\theta}}\, e^{-t^2 e^{-i\theta}/4c\rho}$$

$$= \frac{t^{2n}}{4^n c^n \rho^{2n+\frac{1}{2}}}\, e^{-t^2 \cos\theta/4c\rho + i[t^2 \sin\theta/4c\rho - (2n+\frac{1}{2})\theta]}.$$

Since h is small compared with x, we may write x for ρ, h/x for $\cos\theta$, and 1 for $\sin\theta$. Hence, making this substitution, taking the real part and omitting the $4^n c^n$ in the denominator, we obtain

$$\frac{t^{2n}}{x^{2n+\frac{1}{2}}} e^{-t^2h/4cx^2} \cos\left\{t^2/4cx - (2n+\tfrac{1}{2})\theta\right\},$$

where $\theta = \tan^{-1} x/h = \frac{1}{2}\pi$. Since $t^2/4c$ is of the order of $1{\cdot}25 \times 10^5$, the first term in the argument of the cosine varies very much more rapidly than the second. Disregard the second, therefore, and we obtain finally

$$\frac{t^{2n}}{x^{2n+\frac{1}{2}}} e^{-t^2h/4cx^2} \cos(t^2/4cx) \quad \text{or} \quad \mathrm{A}\cos(t^2/4cx). \tag{4}$$

We shall now find the manner in which the energy is distributed over the different wave-lengths for a given value of t. The wave-length at x is given by

$$\frac{t^2}{4c}\left(\frac{1}{x} - \frac{1}{x+\lambda}\right) = 2\pi;$$

i.e.

$$\frac{t^2\lambda}{4cx^2} = 2\pi \quad \text{or} \quad \lambda = \frac{8\pi c x^2}{t^2}.$$

It thus increases as the square of the distance from the origin. The energy in a distance dx is proportional to $\mathrm{A}^2 dx$. The change of wave-length in this distance is given by

$$d\lambda = d\left(\frac{8\pi c x^2}{t^2}\right) = \frac{16\pi c x\, dx}{t^2}.$$

The energy per wave-length is therefore

$$\mathrm{A}^2 \frac{dx}{d\lambda} = \frac{t^{4n}}{x^{4n+1}} e^{-t^2h/2cx^2} \frac{t^2}{16\pi c x} = \left(\frac{t^2}{x^2}\right)^{2n+1} \frac{1}{16\pi c} e^{-(h/2c)(t^2/x^2)}.$$

Substitute $t^2/x^2 = 8\pi c/\lambda$; then the energy per wave-length is

$$\frac{(8\pi c)^{2n+1}}{\lambda^{2n+1}} \frac{1}{16\pi c} e^{-(h/2c)(8\pi c/\lambda)} = \frac{(8\pi c)^{2n}}{2} \frac{1}{\lambda^{2n+1}} e^{-4\pi h/\lambda}.$$

Wien's law for black body radiation is stated as

$$\frac{c_1}{\lambda^5} e^{-c_2/\lambda \mathrm{T}}.$$

If n is put $= 2$, the two expressions agree. Hence the second of our initial pulses must be of exactly the same nature as that emitted by a black body. What is more, our symbol h varies inversely as T, the absolute temperature of the black body. The unexpected simplicity of the relation seems to promise well for the future of this way of regarding black body radiation.

As a check upon the analysis we can calculate the energy in the initial pulse. It is given in the general case by

$$\int_{-\infty}^{+\infty}\left(\frac{\cos\left(n+\frac{1}{2}\right)\theta}{\rho^{n+\frac{1}{2}}}\right)^2 dx.$$

Now, $x = h\tan\theta$; therefore, $dx = h\sec^2\theta\, d\theta$. Also, $\rho = h\sec\theta$. On substitution the integral becomes

$$\frac{2}{h^{2n}}\int_0^{\pi/2}\cos^2\left(n+\tfrac{1}{2}\right)\theta\cos\theta^{2n-1}\, d\theta.$$

If $n = 2$, this varies as the inverse fourth power of h or the direct fourth power of T. We have thus arrived at Stefan's law of black body radiation.

Let us now definitely make $n = 2$ and examine the shape of the pulse when its maximum has travelled 2 cm. from the origin. For the maximum

$$\frac{d}{dx}\frac{1}{x^{9/2}}e^{-t^2h/4cx^2} = 0,$$

which gives $t = x\sqrt{(9c/h)}$. We see incidentally that the group velocity is $\sqrt{(h/9c)}$. Write 2 for x in this expression for t and substitute the result in (4), at the same time putting $n = 2$. If we omit the constant factor, (4) then becomes

$$\frac{1}{x^{9/2}}e^{-9/x^2}\cos(9/hx).$$

This expression gives the disturbance as a function of x, and it is represented in fig. 2. To make the figure clear h has been given the very large value of $\frac{1}{15}$.

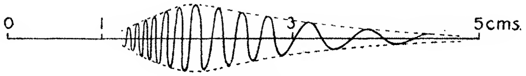

Fig. 2.

It will be seen from fig. 2 that the initial pulse has been dispersed by the medium into a train of waves all comprised between 1 and 5 cm., the long waves coming first and the wave-length gradually decreasing as we get to the rear of the train. There is, of course, a similar train travelling the other way at the same distance on the other side of the origin. The wave-length at the point of maximum amplitude, *i.e.* the dominant wave-length, is obtained by combining

$$\lambda = \frac{8\,\pi c x^2}{t^2} \quad \text{and} \quad t = x\sqrt{(9\,c/h)}.$$

It is therefore $\frac{8}{9}\pi h$ and is always the same, *i.e.* the same wave-length always rides on the crest of the group. In the case represented in fig 2, the dominant wave-length is $8\pi/(9\times 15) = 0{\cdot}186$ cm., a very long heat wave. If it were green, at that distance out there should be roughly 3500 times as many wave-lengths in the train, but, of course, the number always increases the further the group goes.

The period of a wave τ is given by

$$\frac{1}{4\,cx}\left\{(t+\tau)^2-t^2\right\} = 2\,\pi,$$

i.e.

$$\frac{2\,t\tau}{4\,cx} = 2\,\pi \quad \text{or} \quad \tau = \frac{4\,\pi cx}{t}$$

Hence the wave-velocity, v, is

$$\frac{\lambda}{\tau} = \frac{8\,\pi cx^2}{t^2}\,\frac{t}{4\,\pi cx} = \frac{2\,x}{t} = \frac{4\pi cx}{t}\,\frac{1}{2\pi c} = \frac{\tau}{2\,\pi c},$$

the same value as was obtained by substitution in equation (1).

The dispersive power of a medium is usually specified by

$$\frac{\mu_{\mathrm{F}}-\mu_{\mathrm{C}}}{\mu_{\mathrm{D}}-1}.$$

In our hypothetical medium the velocity varies as the wave-length, and the velocity for $\lambda = 5\times 10^{-5}$ is 3×10^{10} cm./sec. Consequently μ_{D} is not far from 1, and it seems more reasonable to measure the dispersive power simply by $\mu_{\mathrm{F}}-\mu_{\mathrm{C}}$, which has the value

$$\frac{5000}{4851} - \frac{5000}{6563} = 0{\cdot}27.$$

For flint glass $\mu_{\mathrm{F}}-\mu_{\mathrm{C}}$ has the value 0·0193 and for air the value $3{\cdot}1\times 10^{-6}$; hence the dispersive power of our hypothetical medium is roughly 14 times that of flint glass and 10^5 times that of air.

I am indebted to Dr. George Green for suggesting that Kelvin's hydrodynamical solutions might be applied to optics; also for showing me an original derivation of Rayleigh's energy law by the application of group velocity.